**Mechanical Systems Laboratory**
**Integral Control; Introduction to Second Order Systems**

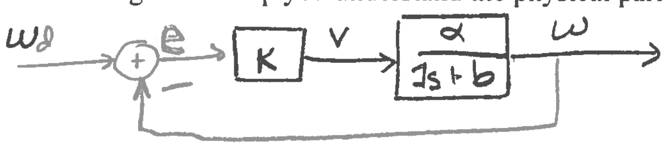
## 1. Integral Control

Imagine that you use proportional feedback to control the velocity of a DC brushed motor:

$$\underbrace{J\dot{\omega} + b\omega}_{\text{dynamics}} = \underbrace{\tau_m = \alpha V}_{\text{current amplifier}} \qquad \underbrace{V = -k(\omega - \omega_d)}_{\text{FB controller}}$$

where v= voltage input to current amplifier that powers the motor, ω = actual angular velocity of motor, sensed with a tachometer, $\omega_d$ = desired angular velocity, K = proportional feedback gain, alpha = proportionality constant relating v (i.e. current amplifier input) to torque output from motor, b = viscous friction.

Draw a block diagram to help you understand the physical parts to the system.




Problem: Show that there is a steady-state error in velocity due to the friction.

Approach 1: Use D.Eq.

$J\dot{\omega} + b\omega = -\alpha K(\omega - \omega_d)$

$J\dot{\omega} + (b + \alpha K)\omega = \alpha K \omega_d$

let $d/dt \to 0$

$$\omega_{ss} = \frac{\alpha K}{b + \alpha K}\omega_d \Rightarrow e_{ss} = \omega_{ss} - \omega_d = \frac{-b}{b + \alpha K}\omega_d$$

Approach 2: Block diagram algebra + LT props.

$$\omega = \frac{\alpha}{Js+b}V = \frac{\alpha}{Js+b}K(\omega_d - \omega)$$

$$\omega\left(\frac{Js+b}{Js+b} + \frac{\alpha K}{Js+b}\right) = \frac{\alpha K}{Js+b}\omega_d$$

$$\omega = \frac{\alpha K}{Js + b + \alpha K}\omega_d$$

let $s \to 0$

$$\omega = \frac{\alpha K}{b + \alpha K}\omega_d$$

KEY IDEA: We can get rid of this steady-state error by using a proportional plus integral (PI) controller:

$$V = -K_p e - K_I \int e\,dt$$

$$J\dot{\omega} + b\omega = -\alpha K_p e - \alpha K_I \int e\,dt$$

$d/dt$

$$J\ddot{\omega} + b\dot{\omega} = -\alpha K_p \dot{e} - \alpha K_I e$$

Note $\ddot{\omega} = \ddot{e}$, $\dot{\omega} = \dot{e}$ if $\omega_d$ = constant

$$J\ddot{e} + (b + \alpha K_p)\dot{e} + \alpha K_I e = 0$$

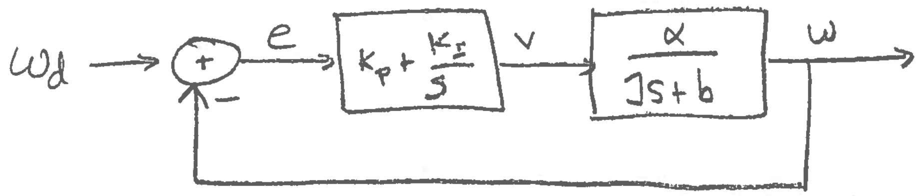
In steady state, $\dot{e} = \ddot{e} = 0$

$$\alpha \cdot K_I e_{ss} = 0 \qquad e_{ss} = 0$$

How does I control work? (try to explain it to your neighbor in words).

Integral control works in the following way:
If error e(t) does not equal zero, then $\int e(t)dt$ increases with time, and eventually the torque (which is proportional to this integral) becomes high enough to overcome friction.

The block diagram for a P-I compensator is:




What is the transfer function for this system?

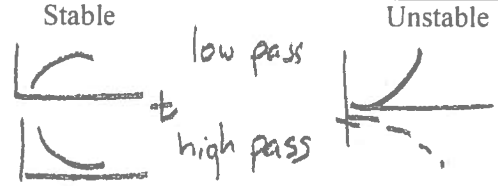
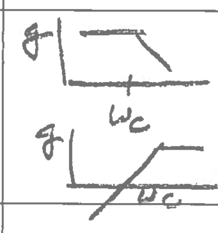
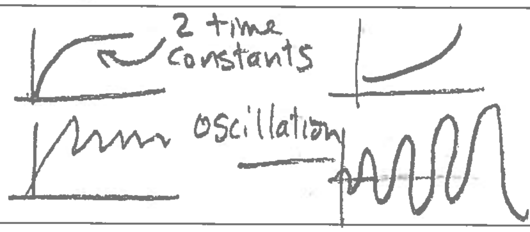
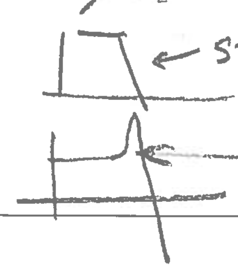
$$\omega = \frac{\alpha}{Js+b} V = \frac{\alpha}{Js+b}\left(\frac{K_p s + K_I}{s}\right)(\omega_d - \omega)$$

$$\omega = \underbrace{\frac{\alpha K_p s + \alpha K_I}{Js^2 + (b + \alpha K_p)s + \alpha K_I}}_{G(s)} \omega_d$$

$$\omega\left(\frac{Js^2 + bs}{Js^2 + bs} + \frac{\alpha K_p s + \alpha K_I}{Js^2 + bs}\right) = \frac{\alpha K_p s + \alpha K_I}{Js^2 + bs} \omega_d$$

$$G(s) = \frac{as + b}{cs^2 + ds + e}$$

This is an example of second order system, which behaves differently than a first order system.

| | Typical behaviors in time domain (step response) | Typical behaviors in frequency domain |
|---|---|---|
| First order system | Stable  low pass + high pass Unstable |  $g$ $\omega_c$ low pass $g$ $\omega_c$ high pass |
| Second order system |  2 time constants oscillation |  steeper cut off resonance |

Important Ideas: integral control can help remove steady state error. However, I-control adds dynamics to the system, which can lead to 2nd order phenomena such as oscillation and resonance.